FRACTAL GEOMETRY AND CHAOS THEORY: FROM OLD PROBLEMS TO NEW MODELS AND METHODS

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I. Abstract

Fractal geometry and chaos theory are deeply rooted in significant problems in the history of mathematics and science. While mathematicians have sought geometrical descriptions of space with its properties, scientists have attempted to characterize the physical properties of fundamental entities present in space and time. These separate investigations frequently influenced each other and led to profound theories, answers, and models. However, at the same time new problems repeatedly arose internal to mathematics and externally in the applications to which mathematics was applied. Fractal geometry issues from these antecedents in response to features and processes in nature not easily represented by historical mathematical models.

II. Introduction

At the foundation of scientific inquiry lies an implicit assumption that some level of inherent orderliness drives all natural processes. Were this not so, there would be little point in scientific investigations that seek patterns in observed phenomena and the codification of such patterns in mathematical formulas. Once observed behaviors and processes have been embedded in a scientific theory and summarized by mathematical "laws", the bane and blessing of such theories and laws are observed pertebations, or deviations, from the patterns that would otherwise be predicted by the mathematical model. These deviations exist as a bane because they challenge the accuracy and correctness of the theory articulated to explain the prior observations. On the other hand, discrepancies from prediction provide the fertile ground needed for theoretical refinements, further inquiry, and alternative explanations.

Some natural events present themselves as more than slight pertebations to the orderly progression that typically characterizes more normative ongoing processes. Chaotic tornados, violent earthquakes, conflagrations in forests and buildings, these and other events contain such turbulent complexities that hope seems to evaporate when it comes to summarizing and formulating these behaviors at a level that would permit predictability. This kind of chaos does not just seem to exist as a more complex but inscrutable level of order. Rather, in these turbulent events, nature does not appear to obey any laws at all. Moreover, and fundamentally more disturbing, in some processes nature seems to slide with ease from a state of order into a state of chaos. Previously rhythmic hearts transition to fibrillation, chemical solutions tip from stability into instability, laminar fluid flows melt into turbulent vortexes, and in all of these predictability gives way to unpredictability. It would seem that chaos and order exist as opposite forces competing in many processes to win the outcome.

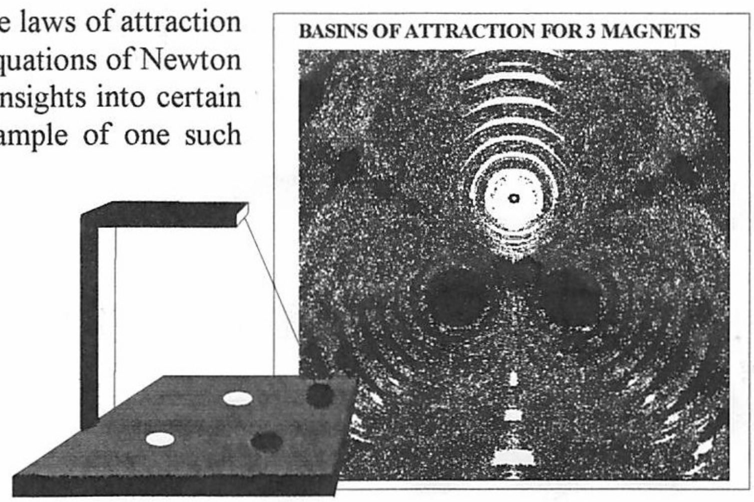
III. An Amazing Historical Progression

In the last quarter of this century significant progress has been made in unlocking the features of chaos itself. Although these achievements have been remarkable, they have derived from a stream of important historical precedents. Ancient efforts by the likes of Ptolemy, Copernicus, and Kepler to describe space with its panoply of stars and planets eventually gave rise to the elaborate theories, techniques, and experiments of Newton, Bernoulli, Coulomb, Einstein, and innumerable others directed at identifying the foundational relationships between both macro and micro objects in terms of force, mass, charge, distance, temperature, energy, and velocity. (See Appendix - The Quest for The Basic Entities and Features Of Nature).

As might be expected, the conception of physical phenomena and their relationships in space were repeatedly refined when reality was observed to significantly deviate from predictions made using existing conventional theories. By way of example, Copernicus abandoned geocentrism as he and others observed that the Ptolemaic assumptions were generating enormous deviations in the prediction of lunar eclipses and the expected positions of planets. Kepler concluded that elliptical orbits better explained planetary motion when his calculations of Mar's orbit disproved circularity. The failure of Newton's predecessors to adequately account for the effect of centripetal force upon planetary orbits caused him to refine Kepler's assumption of a force emanating from the sun into a force formula based upon the masses of the involved bodies. One hundred years later Coulomb succeeded in explaining that the forces that bind atoms together to form molecules, and the forces that bind molecules together to form solids or liquids are electrical rather than the gravitational forces codified by Newton. Unfortunately, the strong Coulomb forces of repulsion in the positively charged atomic nucleus should by themselves cause the nucleus to fly apart. This deviation from the model of electricity and magnetism eventually forced the postulation of the *nuclear force* studied from the days of Einstein onward. If pertebations and problems such as these identified for prior ages were instrumental in provoking scientific advances, then it is problems associated with predicting the behavior of turbulent and wild phenomena in this day that have incited new mathematical and scientific assaults upon the domain of chaos.

IV. The Paradigm of the Magnets

Interestingly, experiments involving the laws of attraction themselves as represented by the force equations of Newton and Coulomb have given remarkable insights into certain important features of chaos. An example of one such experiment is shown in the accompanying illustration. When three equally charged magnets placed at the vertices of an equilateral triangle compete to attract a pendulum suspended over them, the boundaries between the distinct basins of attraction for the three magnets are infinitely frayed. Between a pendulum



release point that eventually leads to a given magnet and a release point that finally leads to a second magnet, there exists interposing points associated with the basin of attraction of the third magnet. The structure of these frayed boundaries are best understood as a Cantor Set in which pairs of points in a given basin of attraction are separated by an intervening region not part of that basin. (See Appendix -Cantor Set Construction).

Surprisingly, the great British mathematician Arthur Cayley anticipated the mathematical possibility of a kind of unpredictable chaotic behavior in the context of Newton's method for finding roots. In an article published in the American Journal of Mathematics (Vol. 2, 1879) Cayley conjectured that the zeros of certain function could compete to attract the sequential approximations of Newton's method in such a way that predicting the zero to which a given initial point might eventually converge could be difficult indeed. The function $f(x) = \frac{1}{(1+x^2)^2} - \frac{25}{36}$ illustrates this behavior to the extent that three points compte to attract the outcome of the Newton process. Magnifications of the boundaries between any two initial points in distinct basins of attraction show that between them are points of the third basin. In these boundary regions, prediction is rendered useless and the simple and orderly algorithm of Newton's method becomes the epitome of unpredictability. An initial point that converges to a particular root lies next to an initial point that converges to the other zero and lying between the two of them is an initial point that diverges to infinity. (See Appendix - Newton's Method Applied to $f(x) = \frac{1}{(1+x^2)^2} - \frac{25}{36}$) A complicating dynamic exists for some functions such as $f(x) = x^{1/3}$ in that the zeros can actually "repel" the successive approximations generated by Newton's method leading to a kind of instability that ultimately causes the sequence to diverge.

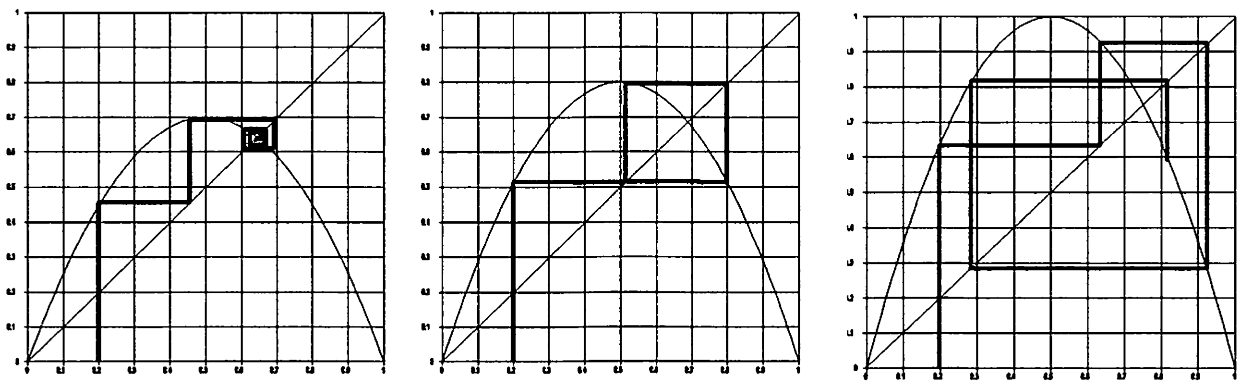
Infinitely frayed boundaries between competing states appear in many mathematical phenomena that are well related to natural processes. For example, applying Newton's method to a simple cubic equation over the complex numbers exhibits three basins of attraction not unlike the basins of the pendulum experiment. The three complex zeros act as the magnets, and the boundaries between their basins of attraction are intricately frayed in a Cantor-like pattern of interposing regions (See Appendix - Newton's Method Applied to $z^3 - 1 = 0$). For some functions, Newton's method can behave with incredible stability leading to a zero with unfailing predictability. For other functions Newton's method can be devastatingly unstable reducing an otherwise orderly algorithm to unfathomable unpredictability.

The attractive gravitational forces of Newton and the attractive and repulsive magnetic/electric forces of Coulomb take on new and heightened meaning as a paradigm for understanding the dynamics of chaos. Like Newton's method iterative processes can be profoundly predictable much like the monotonic and repetitive back and forth swing of a simple pendulum. On the other hand, the same pendulum becomes wildly unpredictable in the presence of even a simple system of three magnets. Incredible progress has been made in unlocking how iterative physical phenomena transition from stability to chaos. In fact, by the late 1970's Mitchell Feigenbaum had completed initial mathematical explorations into he dynamics of such transitions. It is almost startling to realize that Feigenbaum's profound work was performed by iterating initial values through simple quadratic polynomials of the form $f(x) = Ax(1-x)$.

V. Recent Insights into the Features of Chaos

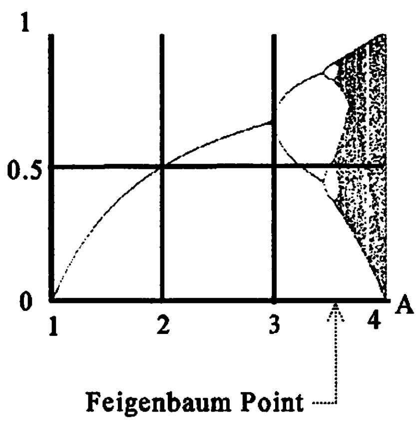
Beginning with an arbitrary initial value x_0, the long-term behavior of the sequence of outcomes $f(x_0)$, $ff(x_0)$, $fff(x_0)$, $ffff(x_0)$, ... produced by repeated composition of $f(x)=Ax(1-x)$ with itself depends completely upon the value of the parameter A. (See Appendix - $f(x) = Ax(1-x)$.)

$f(x) = 2.8x(1-x)$ $\qquad$ $f(x) = 3.2x(1-x)$ $\qquad$ $f(x) = 4x(1-x)$



When A=2.8, the sequence is attracted to $x \approx 0.6428571$ where $y=x$ and $f(x)$ intersect. When A=3.2, two points $x \approx 0.513$ and $x \approx 0.799$ act as a period 2 attractive cycle. However, repeated iteration of an arbitrary initial point through $f(x) = 4x(1-x)$ fails to lead to periodic behavior of any finite cycle. The first two settings of parameter A (specifically 2.8 and 3.2) lead to stable and very predictable behavior under iteration whereas fixing parameter A at 4 leads to rather erratic and very unpredictable behavior. A plot of these various long term behaviors under iteration for each setting of the parameter A between 1 and 4 leads to the Feigenbaum chart shown below.

A close examination of the Feigenbaum map for increasing values of parameter A reveals a sequence of successive "bifurcation" points where the attractive cycle transitions from period 1 to period 2, then from period 2 to period 4, and so on. The accumulated horizontal distances from one bifurcation point to the next approximates a geometric series with ratio converging to 1/4.669. Careful computation of these distances and their sum identifies the so-called Feigenbaum point A=3.5699456 dividing the period doubling regimine on the left from the chaotic region to the right. Setting the paramter A to a value smaller than 3.5699456 implies that iteration through $f(x)=Ax(1-x)$ will eventually stabilize into a repetitive cycle through a finite set of iterates. On the other hand, setting the parameter A to a value greater than 3.5699456 implies that iteration through $f(x)=Ax(1-x)$ will behave in a different and far more erratic manner. Remarkably, many and possibly all turbulent phenomena transition from stability to chaos through the same period doubling pattern of bifurcations represented by the simple class of quadratic functions $f(x)=Ax(1-x)$. Monitoring transition from one level of periodicity to the next in the process of an iterative phenomena provides the possibility of predicting the onset of chaos.



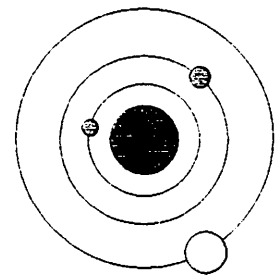
Once transitioned into chaotic state, a number of features are known that further characterize iterative behaviors to the right of the Feigenbaum point. Firstly, distinct initial values selected for iteration can behave in wildly different manners. Some enter periodic patterns while others never repeat. Moreover, infinitely many initial points of both kinds exist throughout the domain and they are utterly intermixed. Secondly, although there are infinitely many inital points that iterate to periodic cycles, their values must be specified with infinite precision since the slightest variation can result in non-periodic behavior. In fact, the values visited repeatedly by a periodic initial point under iteration can act as repellers driving the iterative regimine into non-periodicity and thereby into unpredictable orbits if there is the slightest error in maintaining their values. This sensitivity to slight pertebations makes predicting the behavior of a particular initial value virtually impossible in the absence of infinite precision.

Amazingly, there exist infinitely many narrow windows of stability within the chaotic regimine to the right of the Feigenbaum point. For a given parameter value A within such a window, all initial points from the domain are eventually attracted to the same repetitive cycle dependent only upon the period associated with that particular window. Unfortunately, despite the fact that these windows are numerous, their narrow width within the Feigenbaum map suggest that only a slight variation in the setting of A can trip the iterative behavior from one of incredible stability back into the chaotic regimine that surrounds the window to the right and left. (See Appendix - The Feigenbaum Map).

VI. Some Final Musings

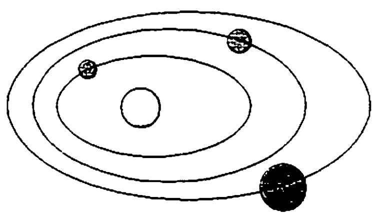
The model of attraction and repulsion implicit in magnet or charged particle experiments as well as in applying Newton's method to certain functions suggests that the dynamics of iterative phenomena may involve competitions between vastly different domains of behavior. Slight changes in the initial state or minor pertebations in the settings of the system may trip the phenomena from one of inherent stabiltiy into one of utter unpredictability and erratic turbulence. The fact that patterns are now understood that signal when a system is transitioning from stability to chaos increasingly raises the possibility that human intervention may be possible in order to prevent or at least ameliorate the effects of turbulent natural events. However, and more fundamentally, if systems present in nature are susceptible to small pertebations, then humankind has a heightened obligation to exercise care and responsible stewardship so as not to upset and despoil beneficial stabilities invested into creation by the Maker.

THE QUEST FOR THE BASIC ENTITIES AND FEATURES OF NATURE



Ptolemy 135 AD GEO-CENTRIC THEORY

The center of the universe is the EARTH. All other heavenly bodies revolve about the earth in circular orbits.

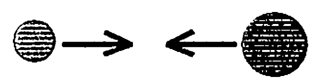


Copernicus 1473-1543 HELIO-CENTRIC THEORY

Kepler 1571-1630

The center of the universe is the SUN. All other heavenly bodies revolve about the sun in elliptical orbits.

Newton 1642-1727 The FORCE of attraction between two bodies depends upon their MASSES and the DISTANCE between them. (1666)



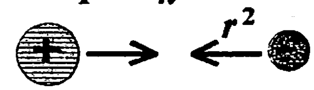
$$F = k \frac{m_1 m_2}{r^2}$$

Bernoulli 1700-1782 The VELOCITY of gas molecules is proportional to the TEMPERATURE. (1738)

$$V \propto k T$$

Coulomb 1736-1806 The FORCE of attraction between two charged particles depends upon the magnitude of the CHARGES and the DISTANCE between them. (1785)

$$F = k \frac{q_1 q_2}{r^2}$$



Einstein 1879-1955 MASS and ENERGY are related by the SPEED of light. (1905)

$$E = M c^2$$

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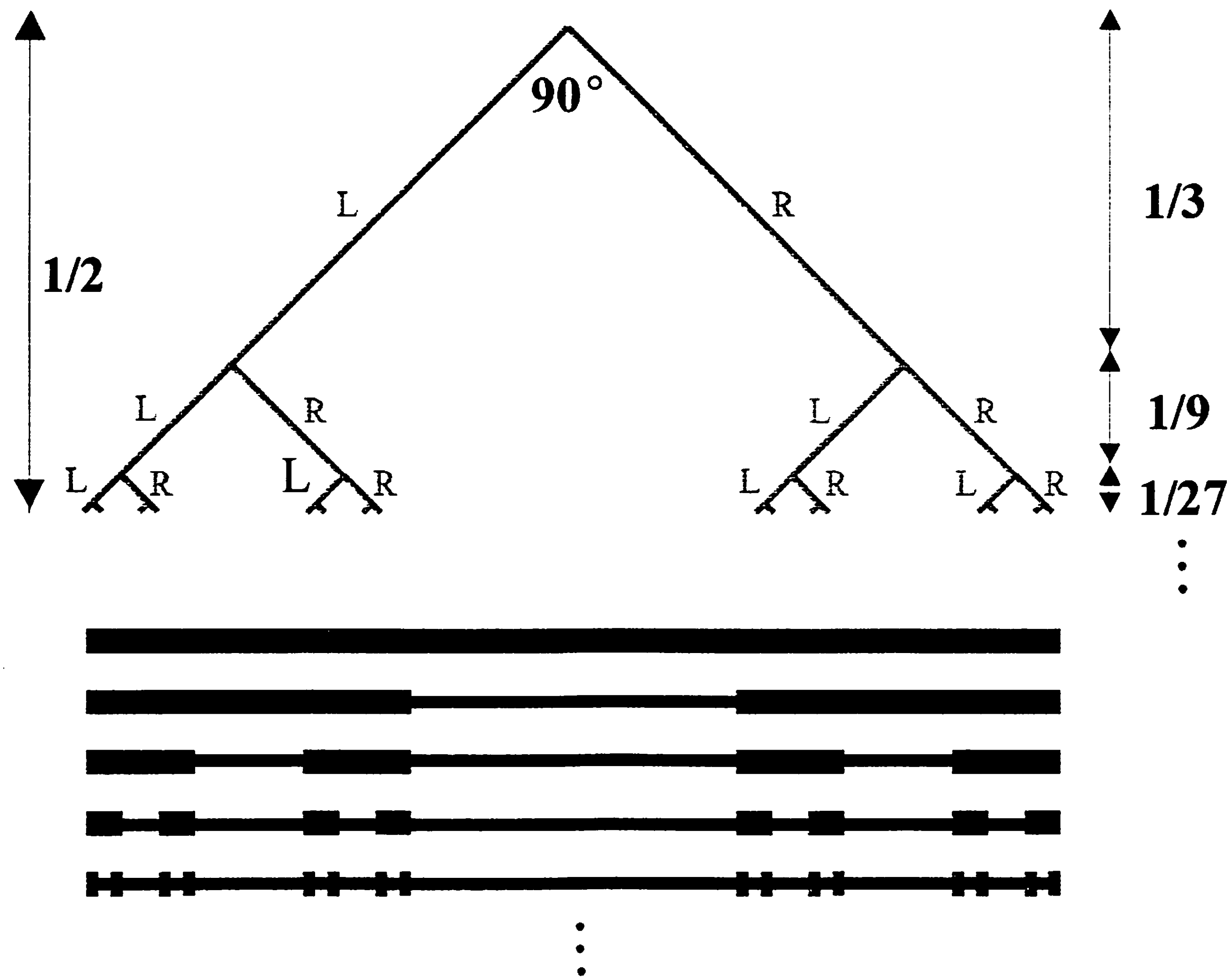
CANTOR SET CONSTRUCTION

A. BINARY TREE LEADING TO ALL POINTS IN CANTOR SET
B. ITERATED REMOVAL OF MIDDLE SUBSEGMENT



Convert every binary decimal in the interval [0,1] into an infinite string of L's and R's by assigning 0 to L, 1 to R. (<u>Example</u> .00110101... LLRRLRLR...)

Then every binary decimal in [0,1] identifies a unique path of L's and R's through the binary tree to a point in the Cantor Set.

THE CANTOR SET MUST CONTAIN AS MANY POINTS AS THERE ARE IN THE INTERVAL [0,1].

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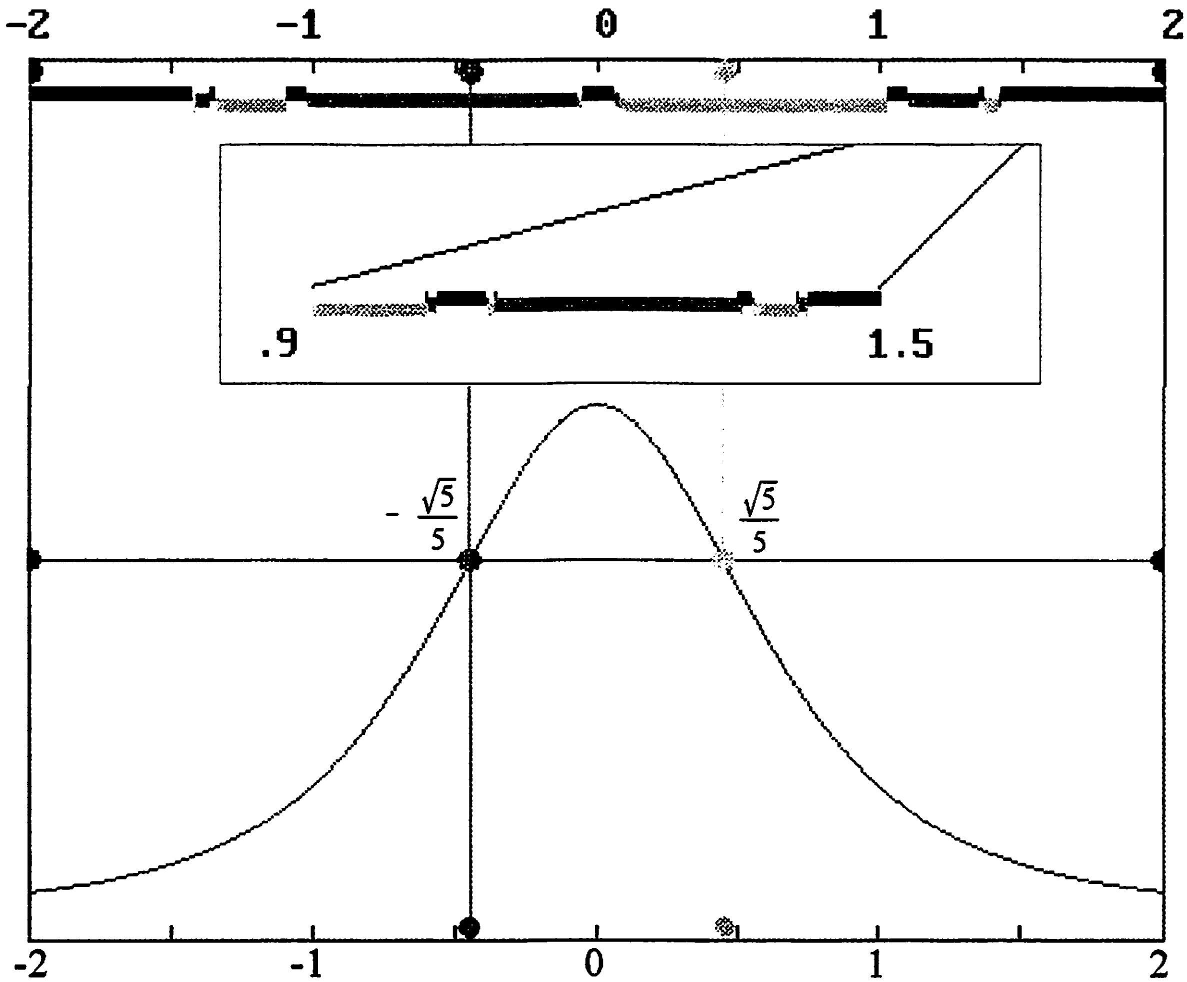
NEWTON'S METHOD APPLIED TO $f(x) = \frac{1}{(1+x^2)^2} - \frac{25}{36}$



ATTRACTORS $x = -\frac{\sqrt{5}}{5}$ $x = \frac{\sqrt{5}}{5}$ ∞

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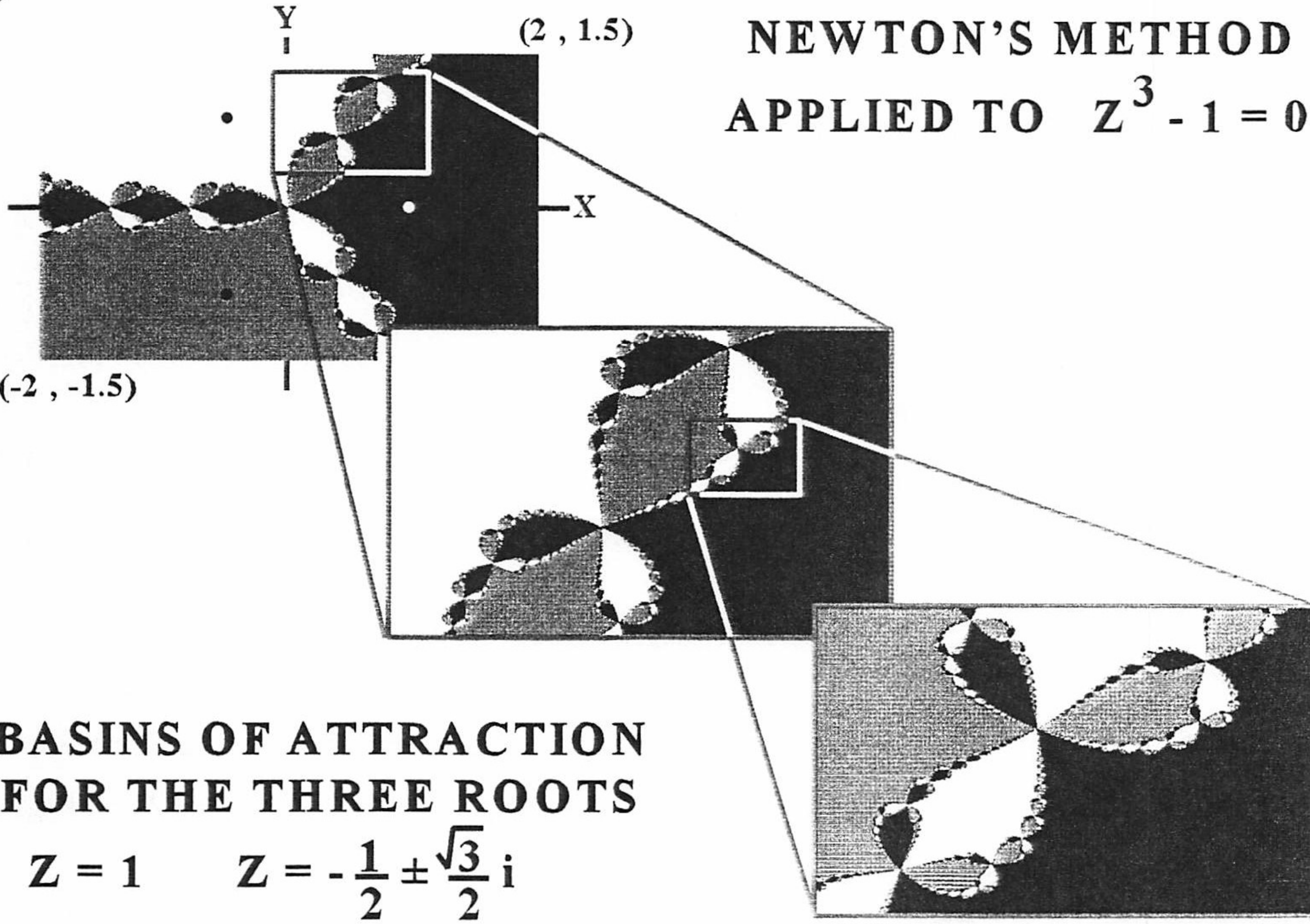
NEWTON'S METHOD APPLIED TO $Z^3 - 1 = 0$



BASINS OF ATTRACTION FOR THE THREE ROOTS

$$Z = 1 \qquad Z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$$

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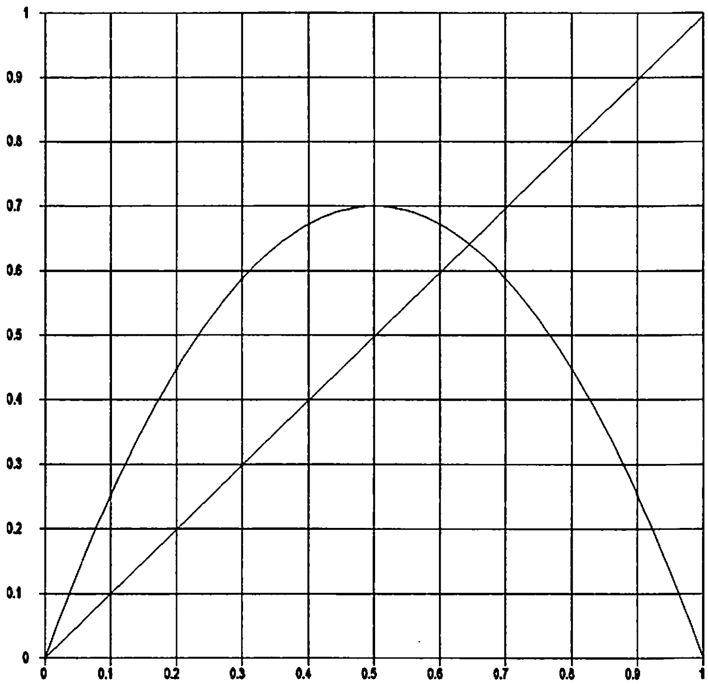
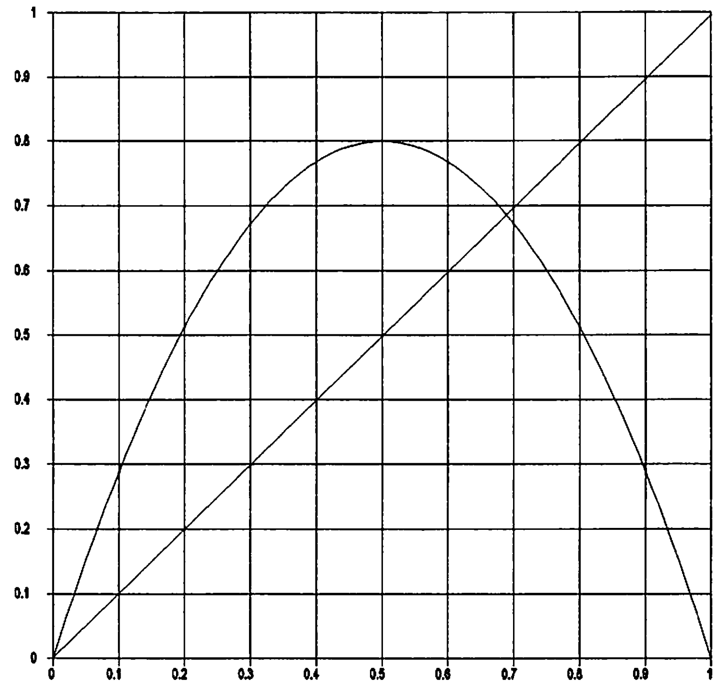
$f(x) = Ax(1-x)$

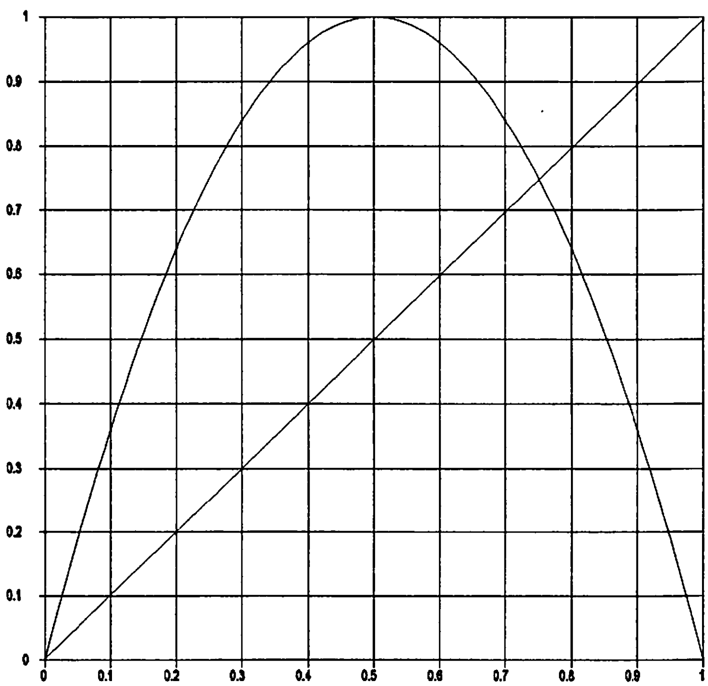
$f(x) = 2.8x(1-x)$



$f(x) = 3.2x(1-x)$



$f(x) = 4x(1-x)$

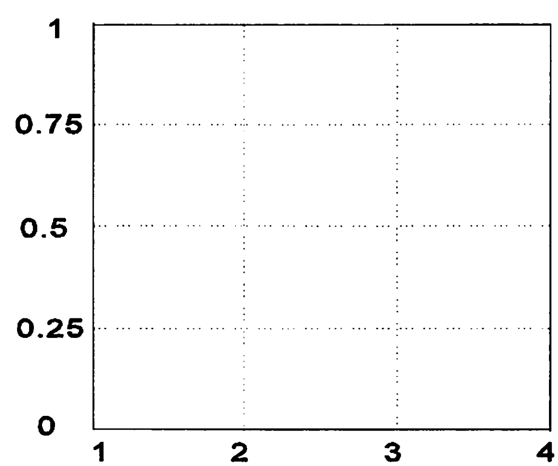


Iterate	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
2.8x(1-x)									

Iterate	0								
3.2x(1-x)									

Iterate	0								
4x(1-x)									

Feigenbaum Chart



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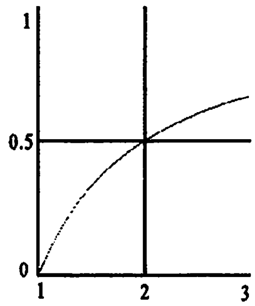
The Feigenbaum Map For f(x) = Ax(1-x)

FEIGENBAUM MAP | **ATTRACTOR** | **REPELLER** | **EXAMPLE**

A=1 to A=3



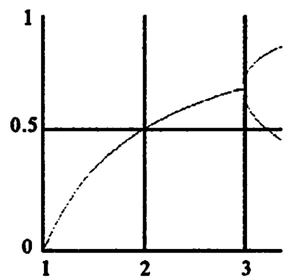
Fixed points of f(x)
Solve Ax(1-x) = x

Attractor: $x = \frac{A-1}{A}$

Repeller: x = 0

A=2.8 Attractor: x = .6428571 Repeller: x = 0

A=1 to A= 3.449489



Fixed points of ff(x)
Solve ff(x) = x

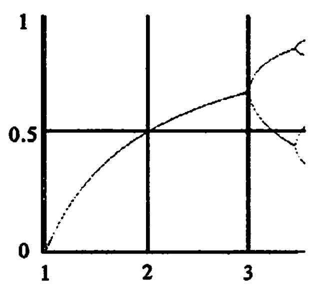
Attractor: $x = \frac{(A+1) \pm \sqrt{A^2-2A-3}}{2A}$

Repeller: x = 0, $x = \frac{A-1}{A}$

A=3.2

Attractor	Repeller
x = .799455	x = 0
x = .513044	x = .6875

A=1 to A=3.544090



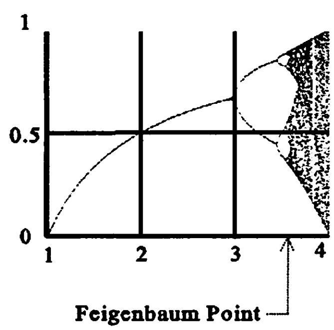
Fixed points of ffff(x)

Repeller: x = 0, $x = \frac{A-1}{A}$, $x = \frac{(A+1) \pm \sqrt{A^2-2A-3}}{2A}$

A=3.5

Attractor	Repeller
x=.50088421	x = 0
x=.38281968	x = .7142857
x=.87499726	x = .428571
x=.82694071	x = .8571429

A=1 to A=4



Feigenbaum Point A=3.5699456 divides Period-Doubling Pattern to the left from the Chaotic Region to the right.

Within the Chaotic Region, narrow Windows of Stability appear.

A=3.73861 Period 5 Window to the right of Feigenbaum Pt.

x=.840357
x=.501559
x=.934643
x=.228373
x=.658814

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